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Comparative study of large-scale Laplacian growth patterns

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We investigate the scaling of cluster size with mass for our simulations of diffusion-limited aggregation (DLA) clusters and dielectric-breakdown (DB) clusters of 10^6 particles grown on a square lattice, and DLA clusters of 10^6 particles grown off lattice. We find that the mass distribution and scaling behavior for the on-lattice DB model are intermediate between those for the on-lattice DLA model and the off-lattice DLA model. We take this as evidence that signatures of lattice anisotropy and the microscopic kinetics of attachment are both manifest at large length scales.

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The diffusion-limited aggregation (DLA) model [1] and the dielectric-breakdown (DB) model [2] are among the most widely studied models for generating fractal growth patterns (for recent reviews, see, e.g., [3,4]). Each of these models is a Laplacian growth process in which diffusing particles released from a distant boundary attach to the perimeter of a growing cluster. The essential difference between the models is a boundary condition [5].

In on-lattice versions the diffusing particle undergoes a random walk on a lattice and particles in the cluster are represented by occupied (aggregate) sites on the lattice. In the DLA model the diffusing particles are terminated at the first surface site (unoccupied site adjacent to an aggregate site) that they contact. This surface site is then converted to an aggregate site. In the DB model the diffusing particles can diffuse through surface sites but are terminated at the first aggregate site they contact. The surface site immediately preceding this termination site in the diffusion process is then converted to an aggregate site. The DB boundary condition introduces an inherent surface tension [6,7]. In this sense, the DB model is more physically appealing than the DLA model, which has zero surface tension [8].

An "off-lattice" Laplacian growth model has also been introduced [9]. In this model, the diffusing particle slides over fixed step lengths in free space until it first intersects the cluster. The diffusing particle is then moved back and is attached at its first point of contact with the cluster. We will follow convention and refer to this growth process as "off-lattice DLA." However, this off-lattice growth process more closely realizes the boundary conditions for the DB model, since the diffusing particle can diffuse along the "surface" of the cluster in each of these models.

Much of the theoretical interest in the above models has been directed at determining precise numerical and algebraic estimates for the fractal dimension D [3,4]. Off-lattice DLA clusters are homogeneous self-similar fractals, with fractal dimension $D \simeq 1.7$ in $d = 2$, independent of cluster size (see, e.g., [10,11]). In contrast, the morphology of on-lattice DLA is size dependent [12–14]. Small clusters appear to be self-similar fractals with $D \simeq 1.7$ [1,8]. Large clusters have an anisotropic star-shaped envelope with the star tips directed along the lattice axes and estimates for D vary in the range $1.5 < D < 1.7$ depending on the size of the cluster [14]. Thus signatures of lattice anisotropy are clearly manifest in large-scale simulations. Small DB clusters also appear to be self-similar fractals with $D \simeq 1.7$ [2,5]. However, we are unaware of any studies of large ($> 10^4$ particles) DB clusters and the on-lattice DLA studies suggest that large clusters are necessary to reveal the full behavior of the model [14].

It is generally believed that the differences between large-scale on-lattice and off-lattice DLA patterns are due to the underlying lattice anisotropy. However, as noted above, the kinetics of attachment implied by the boundary conditions are also different in these two models. The role played by this difference has not been fully explored. Large-scale studies of on-lattice DB patterns are also useful for this purpose because the kinetics of attachment are similar to off-lattice DLA, but the diffusion process is confined to the underlying lattice. The possibility of signatures of the microscopic boundary conditions being manifest at large length scales has also been suggested by the profound difference observed between on-lattice DLA and DB clusters in the zero-noise limit [7].

We have written and implemented computer codes for the growth of large on-lattice DLA and DB clusters and off-lattice DLA clusters. We use the results from these codes to provide estimates for the fractal dimension of DB clusters of up to 10^6 particles and we provide inde-

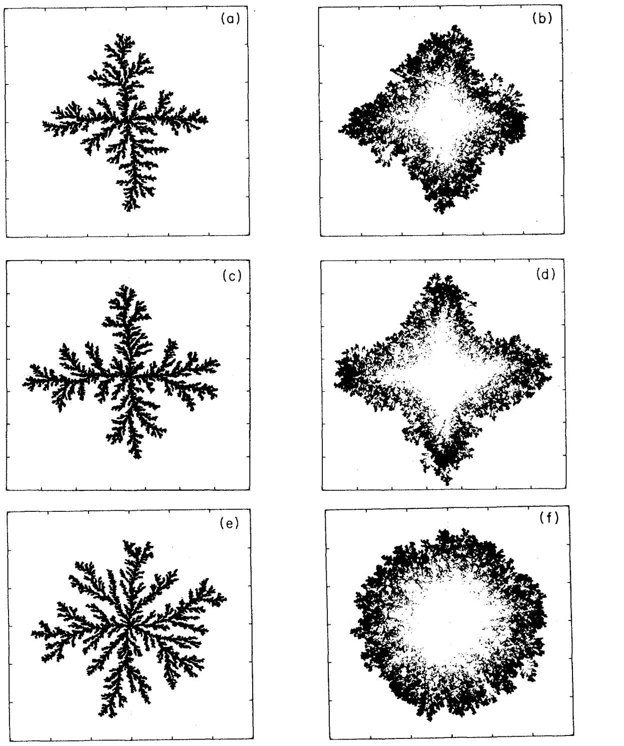

FIG. 1. Representative clusters of size 10^6 for (a) on-lattice DB, (c) on-lattice DLA, and (e) off-lattice DLA. Parts (b), (d), and (f) show the corresponding overlays of the last 10^5 particles added for each of 10 clusters. Note the change of scale for the on-lattice DLA clusters. In each case the spacing between tickmarks is 1000 units.

pendent estimates for the fractal dimension of large on-lattice and off-lattice DLA clusters. Our aim here is not to break existing records for cluster sizes (DLA clusters with more than 10^7 particles have been grown previously, see e.g., [11,15,16]) but to make a comparative study of the large-scale structure of the three Laplacian growth models.

To obtain large DLA clusters it is necessary to move the diffusing particle over a distance greater than a lattice spacing (on lattice) or greater than a particle diameter (off lattice) in a given step of the algorithm. This is done by moving the diffusing particle in one step from its current location to a point on the perimeter of an empty region centered on this current location. To efficiently implement these steps it is necessary to be able to (i) readily identify large empty regions and (ii) readily calculate Laplacian probabilities around the perimeter of the empty region.

In our on-lattice codes we systematically search for the largest empty square and employ a look-up table based on probability formulas derived by McCrea and Whipple [17] to move the particle to the perimeter of the square in one step. Single steps across more than one lattice spacing can also be implemented approximately using the continuum Green's function [13,14] or without approximation using the discrete lattice Green's function [18].

In our off-lattice simulations we store the aggregate particle locations on nested sublattices and sweep through these in sequence from the coarsest grid size to the finest grid size to identify the largest empty circle [13,10,11]. In off-lattice simulations the probability of moving to any given point on the perimeter is uniform around the circle. An additional complication can arise in off-lattice simulations from particle overlaps at the end of a step or during a step. In our off-lattice simulations we implement sliding steps to strictly avoid overlaps. Details of our simulations have been reported elsewhere [19].

We have used the above algorithms to generate 10 clusters of size 10^6 particles for each of the three models; on-lattice DB, on-lattice DLA, and off-lattice DLA. With clusters of this size we can already detect a noticeable anisotropy in a single pattern from the on-lattice simulations [Fig. 1(a) and 1(c)]. An overlay of 10 such patterns for each model [Fig. 1(b), 1(d), and 1(f)] reveals the following clear trends. Both the on-lattice DB and off-lattice DLA models produce patterns that are more compact than the on-lattice DLA model. The off-lattice DLA patterns are essentially isotropic with a circular envelope, whereas the on-lattice DLA patterns have a distinctive cross-shaped envelope [14]. The on-lattice DB patterns are also anisotropic, however the overall envelope appears to be intermediate between a cross and a diamond.

These results indicate that both the effects of lattice anisotropy and boundary conditions are manifest at large scales. The more compact nature of the on-lattice DB and off-lattice DLA patterns reflects the inherent surface tension arising from the similar boundary conditions in these models (recall that on-lattice DLA has no surface

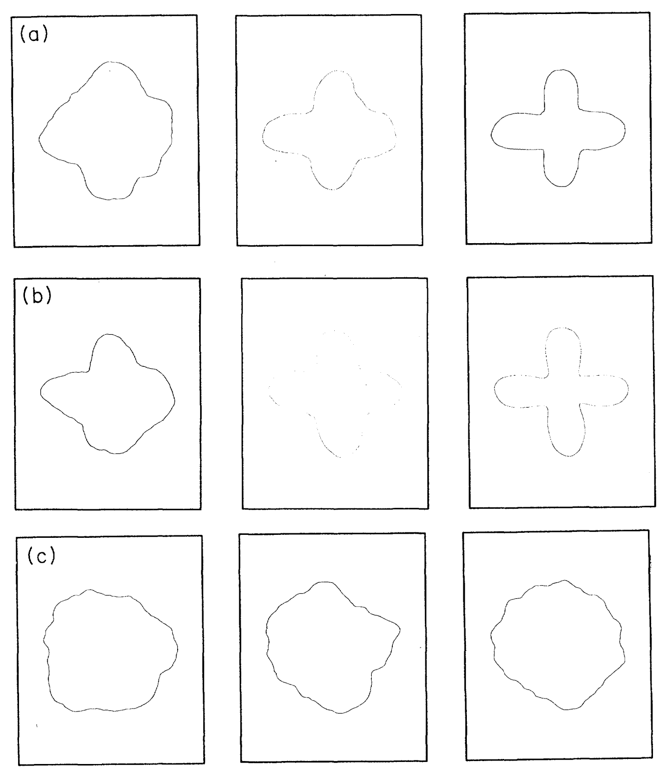

FIG. 2. Angular distribution of mass ($\theta \pm 22.5°$) in Laplacian growth patterns for sizes 10^4, 10^5, and 10^6 particles for (a) on-lattice DB, (b) on-lattice DLA, and (c) off-lattice DLA.

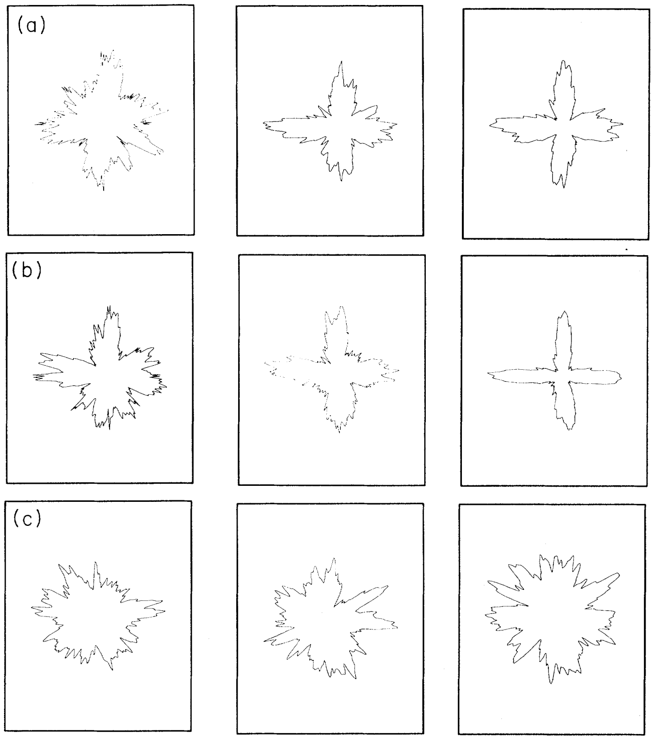

FIG. 3. Angular distribution of mass ($\theta \pm 1^\circ$) in Laplacian growth patterns for sizes 10^4, 10^5, and 10^6 particles for (a) on-lattice DB, (b) on-lattice DLA, and (c) off-lattice DLA.

tension). The envelope of patterns for the on-lattice models reflects the underlying lattice anisotropy. The overall on-lattice DB patterns reflect a balance between the competing effects of lattice anisotropy and surface tension. In the noise-reduced version of this model this competition leads to tip splitting [20,7] and the possibility of a different fractal dimension than that of noise-reduced on-lattice DLA, which does not exhibit tip splitting [20,7].

A useful diagnostic for probing the morphology of Laplacian growth patterns is the angular mass distribution function. For on-lattice DLA a polar plot of this function clearly reveals the transition from a diamond-shaped envelope characteristic of clusters containing about 10^4 particles to a cross-shaped envelope for about 10^6 particles [14]. In Fig. 2 and Fig. 3 we show normalized polar plots of the mass distribution function for clusters containing 10^4, 10^5, and 10^6 particles for each of the three models. Figure 2 is a coarse-grained angular distribution function showing, for each angle θ, the number of aggregate particles within a wedge of $\theta \pm 22.5^\circ$. In Fig. 3, results are shown for a narrower wedge of $\theta \pm 1^\circ$. We have averaged over all 10 clusters in these figures. The coarse-grained angular distributions have an approximate circular shape for the off-lattice DLA clusters over the range of sizes displayed whereas the approximate shape is diamondlike for on-lattice clusters of 10^4 particles and crosslike for on-lattice clusters of $> 10^5$ particles (see also [14] for on-lattice DLA). The ratio of the arm length divided by the arm width is greater in the large-scale on-lattice DLA distributions than it is in the large-scale on-lattice DB distributions. Moreover, the length to width ratio is increasing with cluster size more quickly in the on-lattice DLA distributions. This is especially evident in Fig. 3.

The fractal dimension D of a self-similar cluster can be determined from the scaling relation,

$$M \sim R_g^{1/D}, \tag{1}$$

where M is the number of particles (or mass) of the cluster and R_g is the radius of gyration (see, e.g., [3,4]). For the on-lattice DLA model the fractal dimension is known to depend on the size of the cluster. Thus DLA is only approximately fractal in the sense that the scaling relation (1) is not strictly obeyed. It is, therefore, important to estimate D over a range of scales. We adapt a method developed by Tolman and Meakin [10] to achieve this aim. Data is derived from cluster j by determining $\log_{10} M_i$ and $\log_{10} R_i^j$ (where R is the radius of gyration) at points $M_i = \text{Int}(10^{0.05(i-1)})$ where $i = [21, 101]$. To determine an estimate over a given range of i a linear least squares fit of the data $\log_{10} R_i^j$ vs $\log_{10} M_i$ for $i = p, p+1, \ldots, q$ is performed. The gradient of this line $D_{p,q}^j$ is then averaged over all the clusters to determine $D_{p,q}$. The measure $D_{p,q}$ is the exponent of the power law given by (1) over the range $[p, q]$. If it remains constant over several orders of magnitude it can be termed the fractal dimension.

In Fig. 4 we plot $D_{i-1,i}$ and $D_{i-4,i+4}$ vs $\log_{10} M$ for each of the different models (the latter measurement corresponds to that developed in Ref. [10]). For $M < 10^4$ there are large fluctuations in the value of D due to the relatively small range of M used in the estimations. In the on-lattice simulations the fluctuations are about an approximate median value of $D = 1.72$ whereas in the off-lattice simulation the value of D decreases almost monotonically to 1.72 at $M = 10^4$. As M increases in the

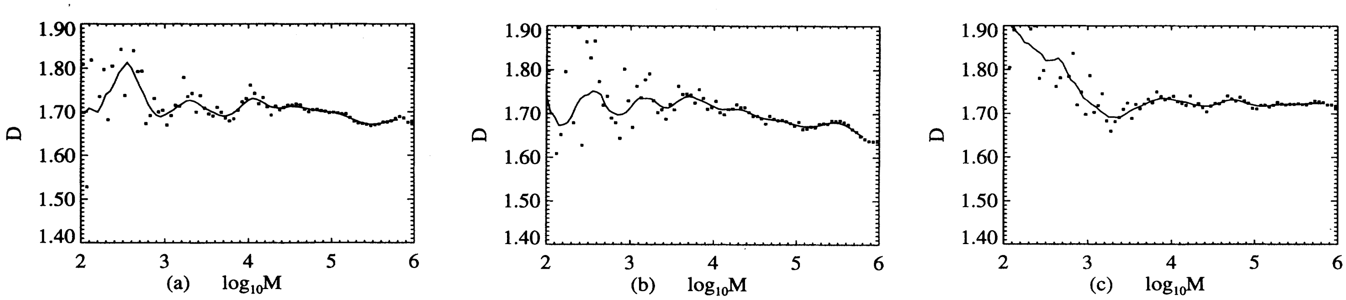

FIG. 4. The scaling exponent D plotted as a function of $\log_{10} M$ for the three different types of Laplacian growth patterns; (a) on-lattice DB, (b) on-lattice DLA, and (c) off-lattice DLA. The symbols indicate the *local* exponent $D_{i,i+1}$ and the solid lines indicate $D_{i-4,i+4}$, a statistically more robust measure.

TABLE I. The values of the fractal exponents for the different classes of clusters. The calculation of $D_{p,q}$ is discussed in the text. The error bars represent 95% confidence limits.

Model	$D_{21,61}$	$D_{61,81}$	$D_{81,101}$	$D_{101,121}$	$D_{41,121}$	$D_{113,121}$
DB	1.793±0.073	1.708±0.024	1.711±0.040	1.678±0.021	1.708±0.013	1.681±0.028
DLA	1.778±0.060	1.723±0.044	1.697±0.019	1.669±0.018	1.704±0.022	1.649±0.030
Off lattice	1.851±0.092	1.708±0.041	1.724±0.026	1.722±0.016	1.731±0.018	1.724±0.016

range $10^4 < M < 10^6$ there is a gradual decrease in the value of D in the on-lattice simulations but D remains essentially constant in the off-lattice simulations. We note too that the rate of decrease is larger for on-lattice DLA than for on-lattice DB.

In Table I, we list $D_{i,j}$ for a range of i, j and error bounds calculated from the standard error among the 10 clusters of each type. Our results for off-lattice clusters are in broad agreement with those reported previously [11]. The off-lattice clusters from our simulations appear to be well-defined self-similar objects. The dimension of $D = 1.724 \pm 0.016$ is slightly higher than that reported elsewhere ($D = 1.712 \pm 0.003$; [11]), but there is agreement within the experimental error bars. Our results for on-lattice DLA clusters are also consistent with earlier results [14]. Our final estimate of $D = 1.649 \pm 0.030$ for the on-lattice DLA model is slightly higher than the value $D \simeq 1.64$ suggested by Meakin *et al.* [14]. Finally we note that for large DB clusters ($M > 5 \times 10^5$) the dimension is intermediate between that of off-lattice DLA and on-lattice DLA. Moreover from Fig. 4, it appears that at $M = 10^6$ the fractal dimension of DB ($D = 1.681 \pm 0.28$) is at or is very close to its asymptotic value.

The main purpose of this paper was to initiate studies of the large-scale structure of the DB model. We compared the growth patterns on the square lattice with on- and off-lattice DLA patterns. Our essential findings were that the angular mass distribution and the scaling of cluster size (radius of gyration) with mass (number of particles) for the DB clusters was intermediate between that for the on-lattice DLA model and the off-lattice DLA model. The on-lattice DLA model is subject to lattice anisotropy but no surface tension whereas the off-lattice DLA model has no lattice anisotropy but the microscopic kinetics of attachment define an inherent surface tension. The on-lattice DB model is subject to both lattice anisotropy and an inherent surface tension.

The differences and similarities between the results for the different models suggests that the signatures of lattice anisotropy and the microscopic kinetics of attachment are both manifest at large length scales. It would be most interesting to determine if one effect dominates over the other in the asymptotic limit. The analysis of large-scale clusters presented here together with our zero-noise studies [7] would suggest that this is not the case but much larger clusters (about two orders of magnitude) would be required to unambiguously settle this point.

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